

A NOTE ON QUIVERS WITH SYMMETRIES

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ABSTRACT. We show that the bases of irreducible integrable highest weight module of a non-symmetric Kac-Moody algebra, which is associated to a quiver with a nontrivial admissible automorphism, can be naturally identified with a set of certain invariant Langrangian irreducible subvarieties of certain varieties associated with the quiver defined by Nakajima. In the case of non-symmetric affine or finite Kac-Moody algebras, the bases can be naturally identified with a set of certain invariant Langrangian irreducible subvarieties of a particular deformation of singularities of the moduli space of instantons over A-L-E spaces.

1. INTRODUCTION

In his remarkable paper [Na1], Nakajima studied among other things the geometry of the moduli space of instantons (for more precise statement, see §2.1.) over A-L-E spaces. He identified a basis of irreducible integrable highest weight module of an affine or finite symmetric Kac-Moody algebra with a set of certain Langrangian irreducible subvarieties of a particular deformation of the moduli space of instantons over A-L-E spaces. By this result, it seems natural (cf. [VW]) to assume that the partition functions of certain twisted N=4 supersymmetric Yang-Mills theory on A-L-E spaces are essentially normalized characters of irreducible integrable highest weight modules of an affine or finite symmetric Kac-Moody algebras (cf.[Kac]). By a well-known results of [KP], these character functions have certain modular properties, i.e., the vector space spanned by the characters of a fixed level admits an action of the modular group, in accordance with S-duality of N=4 supersymmetric Yang-Mills (SYM) theory in 4-dimension. This is the way that the work of [Na1] is thought as evidence in favor of S-duality.

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[Na1] is only concerned with the symmetric Kac-Moody algebras. On the other hand, the results of [KP] also applies to non-symmetric affine Kac-Moody algebras. It is a natural question to ask if one can have a similar treatment of the representations of non-symmetric affine Kac-Moody algebras in terms of the geometry of the moduli space of instantons over A-L-E spaces such that the results of [KP] in the case of non-symmetric affine Kac-Moody algebras can be thought as further evidence in favor of S-duality. This question is the main motivation of our paper.

The idea is to consider quivers with symmetries, i.e., quivers with an admissible non-trivial automorphism (cf. §1.4), a concept due to G.Lusztig in [Lu]. In [Lu], Lusztig associated to each non-symmetric Kac-Moody algebra a quiver with an admissible non-trivial automorphism which is denoted by a . Let us denote by u^a the corresponding non-symmetric Kac-Moody algebra. a induces a natural map between the quiver varieties constructed in [Na1]. By using the results of [Lu] and [K], we will show that certain a -invariant (as a set) irreducible Lagrangian subvarieties of the quiver varieties are in one-to-one correspondence with a basis of irreducible integrable highest weight modules of u^a (cf. Th.3.2.1). This result partially answers the motivating question above (cf. Cor.3.2.2).

Let us describe in more details the content of this paper.

§2 is a preliminary section. We follow closely [Lu] to set up notations. In §2.1 we define the quantized algebra U and its simple integrable module Λ_λ with dominant integral weight λ . §2.2 is a review on quivers, perverse sheaves and canonical bases. In §2.3 we recall the results of [K] which establishes a one-to-one correspondence between the canonical bases and certain irreducible Lagrangian subvarieties. In §2.4 we introduce the concept of admissible automorphisms due to G.Lusztig. Lemma 2.4.1 shows that the one-to-one correspondence in §2.3 is compatible with the action of the admissible automorphisms on the quiver varieties.

In §3.1 we recall the definitions and results of certain modified quiver varieties which are motivated by the study of moduli space of instantons over A-L-E spaces due to Nakajima (cf. [Na1]). Lemma 3.1.1 is essentially the same as Lemma 11.5 of [Na1]. Lemma 3.1.2 is the most important lemma. It identifies a basis of simple integrable module of U with a symmetric Cartan datum with certain irreducible subvarieties as defined in §3.1. In the case of U with a finite symmetric Cartan datum, Lemma 3.1.2 follows from §11 of [Na1].

Th.3.1.3. extends the results in §11 of [Na1] to include all symmetric Cartan datum by using Lemma 3.1.2.

In §3.2, we consider a quiver with a nontrivial admissible automorphism a . Recall the associated non-symmetric Kac-Moody algebra is denoted by u^a . Th.3.2.1, which identifies the basis of a simple integrable module of u^a with a -invariant (as a set) irreducible Lagrangian subvarieties defined in §3.1, follows from Lemma 2.4.1, Lemma 3.1.2 and the results of [Lu].

Cor.3.2.2 follows from Th.3.2.1 in the special case of affine quivers. The modular properties of naturally defined functions in Cor.3.2.2. (cf. [KP]) suggests that these functions should be identified with the partition function of certain twisted N=4 SYM theory on A-L-E spaces together with an admissible automorphism, in accordance with S-duality. Cor.3.2.2. thus partially answer the question raised at the beginning of this introduction.

In §3 we concludes with questions and speculations related to string-string duality. For a finite set A , we shall use $\sharp A$ to denote the number of elements in the set A in this paper.

2. PRELIMINARIES

2.1. Definition of U. A **Cartan datum** is a pair (I, \cdot) consisting of a finite set I and a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$, with values in \mathbb{Z} . It is assumed that: (a) $i \cdot i \in 2, 4, 6, \dots$ for any $i \in I$;

(b) $2\frac{i \cdot j}{i \cdot i} \in 0, -1, -2, \dots$ for any $i \neq j$ in I .

Two Cartan data (I, \cdot) and (I, \circ) are said to be **proportional** if there exist integers $a, b \geq 1$ such that $ai \circ j = bi \cdot j, \forall i, j \in I$.

A Cartan datum (I, \cdot) is said to be **symmetric** if $i \cdot i = 2, \forall i \in I$.

A Cartan datum (I, \cdot) is said to be **simply laced** if it is symmetric and $i \cdot j \in \{0, -1\}, \forall i \neq j \in I$.

A Cartan datum (I, \cdot) is said to be **irreducible** if I is non-empty and for any $i \neq j$ in I there exists a sequence $i = i_1, i_2, \dots, i_n = j$ in I such that $i_p \cdot i_{p+1} < 0$ for $p = 1, 2, \dots, n-1$.

A Cartan datum (I, \cdot) is said to be of **finite type** if the symmetric matrix $(i \cdot j)$ indexed by $I \times I$ is positive definite.

A Cartan datum (I, \cdot) is said to be of **affine type** if it is irreducible and the symmetric matrix $(i \cdot j)$ indexed by $I \times I$ is positive semi-definite, but not positive definite.

A **root datum** of type (I, \cdot) consists, by definition, of

(a) two finitely generated free abelian groups Y, X and a perfect bilinear pairing $\langle, \rangle : Y \times X \rightarrow \mathbb{Z}$;

(b) an embedding $I \subset X (i \rightarrow i')$ and an embedding $I \subset Y (i \rightarrow i)$ such that:

$$(c) \langle i, j' \rangle = 2 \frac{i \cdot j}{i \cdot i}.$$

In particular, we have (d) $\langle i, i' \rangle = 2$ for all i ;

$$(e) \langle i, j' \rangle \in 0, -1, -2, \dots \text{ for } i \neq j.$$

Thus $(\langle i, j' \rangle)$ is a symmetrizable generalized Cartan matrix. The imbeddings (b) induce homomorphisms $\mathbb{Z}[I] \rightarrow Y, \mathbb{Z}[I] \rightarrow X$; we shall often denote, again by ν , the image of $\nu \in \mathbb{Z}[I]$ by either of these homomorphisms.

We shall denote by u the universal enveloping algebra of the Kac-Moody algebra g associated with the symmetrizable generalized Cartan matrix above (cf.[Kac]).

A root datum above is said to be X -regular (resp. Y -regular) if the image of the embedding $I \subset X$ is linearly independent in X (resp. the image of the embedding $I \subset Y$ is linearly independent in Y). In this paper we shall be concerned **only** with a root datum which is both X -regular and Y -regular. Such a root datum always exists by §2.2.2 of [Lu]. We shall choose such a root datum for each Cartan datum.

Let v be an indeterminate. For any $i \in I$, we set $v_i = v^{i \cdot i/2}$. Assume that a Cartan datum (I, \cdot) is given.

Let $(Y, X, \langle, \rangle, \dots)$ be the chosen root datum of type (I, \cdot) as above.

Definition 2.1.1. U is the associative algebra over the field $\mathbb{Q}(v)$ of rational functions of v with generators $E_i, F_i, K_\mu, i \in I, \mu \in Y$ and the following defining relations:

$$\begin{aligned} K_0 &= 1, K_\mu K_{\mu'} = K_{\mu+\mu'} \\ K_\mu E_j K_\mu^{-1} &= v^{\langle i, \mu \rangle} E_j, \quad K_\mu F_j K_\mu^{-1} = v^{-\langle i, \mu \rangle} F_j \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}} \\ \sum_{r+r'=1-\langle i, j' \rangle} (-1)^r (X_i^\pm)^{(r)} X_j^\pm (X_i^\pm)^{(r')} &= 0 \quad \text{if } i \neq j. \end{aligned}$$

Here we used notations $X_i^+ = E_i, X_i^- = F_i, \tilde{K}_{\pm i} = K_{\pm(i \cdot i/2)i}$. For any integer $r \geq 0$, and for $1 \leq i \leq n$, define $(E_i)^{(r)} = \frac{(E_i)^r}{[r]_v!}$, $(F_i)^{(r)} = \frac{(F_i)^r}{[r]_v!}$ where the v -factorial $[r]_v!$ is defined by $[r]_v! = [r]_v \cdot [r-1]_v \cdots [1]_v$.

It is shown in Cor.33.1.5 of [Lu] that the above definition in terms of generators and relations coincides with the definition in Chap.3 of [Lu].

Denote by U^- the $\mathbb{Q}(v)$ subalgebra of U generated by $F_i, \forall i \in I$.

For $\nu \in \mathbb{N}[I]$, we denote by U_ν^- the finite dimensional $\mathbb{Q}(v)$ -subspace of U^- spanned by the monomials F_{i_1}, \dots, F_{i_r} such that for any $i \in I$, the number of occurrences of i in the sequence i_1, \dots, i_r is equal to ν_i . U_ν^- is defined in a similar way.

Define a category C as follows. An object of C is a U -module M with a given direct sum decomposition $M = \oplus_{\lambda \in X} M^\lambda$ as a $\mathbb{Q}(v)$ vector space such that $K_i m = v^{\langle \mu, \lambda \rangle} m$ for any i, λ and $m \in M^\lambda$.

An object $m \in C$ is said to be integrable if for any $m \in M$ and i , there exists $n_0 \geq 1$ such that $E_i^{(n)} m = F_i^{(n)} m = 0$ for all $n \geq n_0$. Denote by U^- (resp. ${}_A U^-$) the subalgebra of U (resp. ${}_A U$) generated by $F_i, K_i^{\pm 1}$ (resp. $F_i^{(r)}, K_i^{\pm 1}$). The Verma module M_λ is defined to be U/J where $J = \sum_i U E_i + \sum_\mu U(k_\mu - v^{\langle \mu, \lambda \rangle})$ is the left ideal of U . We define $X^+ = \{\lambda \in X \mid \langle i, \lambda \rangle \in \mathbb{N}, \forall i\}$. The elements of X^+ are called dominant integrable weights.

Let I_λ be the left ideal in U^- generated by the elements $F_i^{\langle i, \lambda \rangle + 1}$. Then $\Lambda_\lambda = U^- / I_\lambda$ is an integrable simple module of U (see Prop. 3.5.6 of [Lu]).

Let η_λ be the unique (up to a constant) highest vector in Λ_λ . For $\nu \in \mathbb{N}[I]$, we define $\Lambda_\lambda(\nu)$ to be the subspace of Λ_λ spanned by $U_\nu^- \cdot \eta_\lambda$. It is clear that $\Lambda_\lambda(\nu)$ is the weight space of Λ_λ with weight $\lambda - \nu$. Remember that we have used the same ν to denote its image in X^+ as explained above (after the definition of root datum).

We can define the simple integrable module $L(\lambda)$ of u with dominant integrable weight λ similarly as above. Let η'_λ be the highest vector in $L(\lambda)$. For $\nu \in \mathbb{N}[I]$, we define $L(\nu, \lambda)$ to be the subspace of $L(\lambda)$ spanned by $u_\nu^- \cdot \eta'_\lambda$.

2.2. Quivers, perverse sheaves and canonical bases. By definition, a finite **graph** is a pair consisting of two finite sets I (vertices) and H (edges) and a map which to each $h \in H$ associates a two-element subset $[h]$ of I .

Suppose a finite graph is given. In this graph, two different vertices may be joined by several edges, but any vertex is not joined with itself by any edges. Suppose two maps, $H \rightarrow I$ denoted by $h \rightarrow \text{out}(h)$, $H \rightarrow I$ denoted by $h \rightarrow \text{in}(h)$, and an involution $h \rightarrow \bar{h}$ are given. We assume that they satisfy the following conditions:

$$\text{in}(\bar{h}) = \text{out}(h), \text{out}(\bar{h}) = \text{in}(h)$$

, and $\text{out}(h) \neq \text{in}(h)$ for all $h \in H$.

An **orientation** of the graph is a choice of a subset $\Omega \subset H$ such that

$$\Omega \cup \bar{\Omega} = H, \Omega \cap \bar{\Omega} = \emptyset$$

A **quiver** is a graph with an orientation.

To a graph (I, H) we associate a Cartan datum as follows: for any $i, j \in I, i \neq j$, we define $i \cdot j = \#\{h \in H ; \text{out}(h) = i, \text{in}(h) = j\}$. Notice that such a Cartan datum is always symmetric.

We denote by u the universal enveloping algebra of the corresponding Kac-Moody Lie algebra and U the quantized algebra associated to the root datum. Let \mathcal{V} be the family of I -graded complex vector spaces $V = \oplus_{i \in I} V_i$.

For each $V \in \mathcal{V}$, $\dim V := \sum_i \dim V_i i \in \mathbb{N}[I]$ is called the dimension vector. By abuse of notation, if $V \in \mathcal{V}$ and we write $V \in \mathbb{N}[I]$, it is assumed that we have denoted $\dim V$ by V .

For $\nu = \sum_i \nu_i i \in \mathbb{N}[I]$, we shall sometimes identify ν as a dominant integral weight, denoted again by ν , such that $\langle i, \nu \rangle = \nu_i$. In this way, $\dim V$ can be identified as a dominant integral weight.

For $\nu \in \mathbb{N}[I]$, let \mathcal{V}_ν be the family of I -graded complex vector spaces V with $\dim V = \nu$.

Let us define the complex vector spaces $E_{V, \Omega}$ and X_V by

$$\begin{aligned} E_{V, \Omega} &= \bigoplus_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}), \\ X_V &= \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}). \end{aligned}$$

In the sequel, a point of $E_{V, \Omega}$ or X_V will be denoted as $B = (B_h)$. Here B_h is in $\text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)})$.

We define the symplectic form ω on X_V by

$$\omega(B, B') = \sum_{h \in H} \varepsilon(h) \text{tr}(B_{\bar{h}} B'_h),$$

where $\varepsilon(h) = 1$ if $h \in \Omega$, $\varepsilon(h) = -1$ if $h \in \bar{\Omega}$. We sometimes identify X_V and the cotangent bundle of $E_{V, \Omega}$ via ω .

The group $G_V = \prod_{i \in I} GL(V_i)$ acts on $E_{V, \Omega}$ and X_V by

$$G_V \ni g = (g_i) \quad : \quad (B_h) \mapsto (g_{\text{in}(h)} B_h g_{\text{out}(h)}^{-1}),$$

where $g_i \in GL(V_i)$ for each $i \in I$.

The Lie algebra of G_V is $\mathfrak{g}_V = \bigoplus_{i \in I} \text{End}(V_i)$. We denote an element of \mathfrak{g}_V by $A = (A_i)_{i \in I}$ with $A_i \in \text{End}(V_i)$. The infinitesimal action of $A \in \mathfrak{g}_V$ on X_V at $B \in X_V$ is given by $[A, B]$. Let $\mu : X_V \rightarrow \mathfrak{g}_V$ be the moment map associated with the G_V -action on the symplectic vector space X_V . Its i -th component $\mu_i : X_V \rightarrow \text{End}(V_i)$ is given by

$$\mu_i(B) = \sum_{\substack{h \in H \\ i = \text{out}(h)}} \varepsilon(h) B_{\bar{h}} B_h.$$

For a non-negative integer n , we set

$$\mathfrak{S}_n = \{\sigma = (h_1, h_2, \dots, h_n) ; h_i \in H, \text{in}(h_1) = \text{out}(h_2), \dots, \text{in}(h_{n-1}) = \text{out}(h_n)\},$$

and set $\mathfrak{S} = \bigcup_{n \geq 0} \mathfrak{S}_n$. For $\sigma = (h_1, h_2, \dots, h_n)$, we set $\text{out}(\sigma) = \text{out}(h_1)$, $\text{in}(\sigma) = \text{in}(h_n)$. For $B \in X_V$ we set $B_\sigma = B_{h_n} \cdots B_{h_1} : V_{\text{out}(h_1)} \rightarrow V_{\text{in}(h_n)}$. If $n = 0$, we understand that $\mathfrak{S}_n = \{1\}$ and B_1 is the identity. An element B of X_V is called nilpotent if there exists a positive integer n such that $B_\sigma = 0$ for any $\sigma \in \mathfrak{S}_n$.

Definition 2.2.1. We set

$$X_{0V} = \{B \in X_V ; \mu(B) = 0\}$$

and

$$\Lambda_V = \{B \in X_V ; \mu(B) = 0 \text{ and } B \text{ is nilpotent}\}.$$

It is clear that Λ_V is a G_V -stable closed subvariety of X_V . It is known that Λ_V is a Lagrangian variety (cf. Remark (1) of 5.11 in [Na1]).

Let us recall the results of Lusztig on canonical bases. We write $D(X)$ for the bounded derived category of complexes of sheaves of \mathbb{C} -vector spaces on the associated complex variety X over \mathbb{C} . Objects of $D(X)$ are referred to as complexes. We shall use the notations of [BBD]; in particular, $[d]$ denotes a shift by d degrees, and for a morphism f of algebraic varieties, f^* denotes the inverse image functor, $f_!$ denotes direct image with compact support, etc.

We fix an orientation Ω of quiver. Let $\nu \in \mathbb{N}[I]$ and let S_ν be the set of all pairs (i, a) where $i = (i_1, \dots, i_m)$ is a sequence of elements of I and $a = (a_1, \dots, a_m)$ is a sequence of non-negative integers such that $\nu_i = \sum_{l, i_l = i} a_l$. Let $(i, a) \in S_\nu$. A flag of type (i, a) is, by definition, a sequence $\phi = (V = V^0 \supset V^1 \supset \dots \supset V^m = 0)$ of I -graded subspace of V such that, for any $l = 1, 2, \dots, m$, the I -graded vector space V^{l-1}/V^l is zero in degree $\neq i_l$ and has dimension a_l in degree i_l . We define a variety $\tilde{\mathcal{F}}_{i,a}$ of all pairs (B, ϕ) such that $B \in E_{V,\Omega}$ and ϕ is a B -stable flag of type (i, a) . The group G_V acts on $\tilde{\mathcal{F}}_{i,a}$ in natural way. We denote by $\pi_{i,a} : \tilde{\mathcal{F}}_{i,a} \rightarrow E_{V,\Omega}$ the natural projection. We note that $\pi_{i,a}$ is a G_V -equivariant proper morphism. We set $L_{i,a;\Omega} = (\pi_{i,a})_!(1) \in \mathcal{D}(\mathcal{E}_{V,\Omega})$. Here $1 \in \mathcal{D}(\tilde{\mathcal{F}}_{i,a})$ is the constant sheaf on $\tilde{\mathcal{F}}_{i,a}$. By the decomposition theorem [BBD], $L_{i,a;\Omega}$ is a semisimple complex. Let $\mathcal{P}_{V,\Omega}$ be the set of isomorphism class of simple perverse sheaves L on $E_{V,\Omega}$ such that $L[d]$ appears as direct summand of $L_{i,a;\Omega}$ for some $(i, a) \in S_\nu$ and some $d \in \mathbb{Z}$. We write $\mathcal{Q}_{V,\Omega}$ for the subcategory of $\mathcal{D}(\mathcal{E}_{V,\Omega})$ consisting of all complexes that are isomorphic to finite direct sums of complexes of the form $L[d]$ for various simple perverse sheaves $L \in \mathcal{P}_{V,\Omega}$ and various $d \in \mathbb{Z}$. Any complex in $\mathcal{Q}_{V,\Omega}$ is semisimple and G_V -equivariant. Take $V \in \mathcal{V}_\nu, V' \in \mathcal{V}_{\nu'}, \bar{V} \in \mathcal{V}_{\bar{\nu}}$ for $\nu = \nu' + \bar{\nu}$ in $\mathbb{N}[I]$. We consider the diagram

$$E_{\bar{V},\Omega} \times E_{V',\Omega} \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_{V,\Omega}. \quad (2.1)$$

Here E' is the variety of $(B, \bar{\phi}, \phi')$ where $B \in E_{V, \Omega}$ and $0 \rightarrow \bar{V} \xrightarrow{\bar{\phi}} V \xrightarrow{\phi'} V' \rightarrow 0$ is a B -stable exact sequence of I -graded vector spaces, and E'' is the variety of (B, C) where $B \in E_{V, \Omega}$ and C is a B -stable I -graded subspace of V with $\dim C = \bar{\nu}$. The morphisms p_1 , p_2 and p_3 are defined by $p_1(B, \bar{\phi}, \phi') = (B|_{\bar{V}}, B|_{V'})$, $p_2(B, \bar{\phi}, \phi') = (B, \text{Im}(\bar{\phi}))$ and $p_3(B, C) = B$. Note that p_1 is smooth with connected fiber, p_2 is a principal $G_{V'} \times G_{\bar{V}}$ -bundle, and p_3 is proper.

Let $L' \in \mathcal{Q}_{V', \Omega}$ and $\bar{L} \in \mathcal{Q}_{\bar{V}, \Omega}$. Consider the exterior tensor product $\bar{L} \boxtimes L'$. Then there is $(p_2)_* p_1^*(\bar{L} \boxtimes L') \in \mathcal{D}(\mathcal{E}'')$ such that $(p_2)^*(p_2)_* p_1^*(\bar{L} \boxtimes L') \cong p_1^*(\bar{L} \boxtimes L')$. We define $L' * \bar{L} \in \mathcal{Q}_{V, \Omega}$ by $(p_3)_!(p_2)_* p_1^*(\bar{L} \boxtimes L')[d_1 - d_2]$ where d_i is the fiber dimension of p_i ($i = 1, 2$).

Let $\mathcal{K}_{V, \Omega}$ be the Grothendieck group of $\mathcal{Q}_{V, \Omega}$. We considered it as a $\mathbb{Z}[q, q^{-1}]$ -module by $q(L) = L[1]$, $q^{-1}(L) = L[-1]$. Then $\mathcal{K}_{\Omega} = \bigoplus_{\nu \in Q_-} \mathcal{K}_{V(\nu), \Omega}$ has a structure of an associative graded $\mathbb{Z}[q, q^{-1}]$ -algebra by the operation $*$. We denote by $F_i \in \mathcal{K}_{V(i), \Omega}$ the element attached to $1 \in \mathcal{D}(\mathcal{E}_{V(i), \Omega})$ where $\dim V(i) = \sum_j \delta_{ij} j$. The following theorem is due to Lusztig (cf. Th.13.2.11 of [Lu]):

Theorem 2.2.2. *There is a unique $\mathbb{Q}(q)$ -algebra isomorphism*

$$\Gamma_{\Omega} : U_q^-(\mathfrak{g}) \rightarrow \mathcal{K}_{\Omega} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q)$$

such that $\Gamma_{\Omega}(f_i) = F_i$.

Let us identify $L \in \mathcal{P}_{V, \Omega}$ with $L \otimes 1 \in \mathcal{K}_{\Omega} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q)$. We set $\mathbf{B} = \Gamma_{\Omega}^{-1}(\bigsqcup_{V \in \mathcal{V}} \mathcal{P}_{V, \Omega})$ and call it the canonical basis of $U_q^-(\mathfrak{g})$.

The canonical bases \mathbf{B} have some remarkable properties. Given $i \in I$, we set $\mathbf{B}_{i, \geq n} = \mathbf{B} \cap F_i^n U^-$ and ${}^{\sigma} \mathbf{B}_{i, \geq n} = \mathbf{B} \cap U^- F_i^n$. Let $\mathbf{B}_{i, n} = \mathbf{B}_{i, \geq n} - \mathbf{B}_{i, \geq n+1}$ and ${}^{\sigma} \mathbf{B}_{i, n} = {}^{\sigma} \mathbf{B}_{i, \geq n} - {}^{\sigma} \mathbf{B}_{i, \geq n+1}$. By 14.3 of [Lu] we have a partition $\mathbf{B}_V = \bigsqcup_{n \geq 0} \mathbf{B}_{V; i; n}$ where $\mathbf{B}_V := \Gamma_{\Omega}^{-1}(\mathcal{P}_{V, \Omega})$ and $\mathbf{B}_{V; i; n} := \mathbf{B}_V \cap \mathbf{B}_{i, n}$. Let $\lambda \in X^+$ be a dominant integral weight and let Λ_{λ} be the U -module as defined in §1.1. Let η_{λ} be the image of $1 \in U^-$ under the quotient map from U^- to Λ_{λ} . Let $\mathbf{B}(\lambda) = \bigcap_{i \in I} (\bigcup_{n; 0 \leq n \leq \langle i, \lambda \rangle} {}^{\sigma} \mathbf{B}_{i, n})$. Then the map $b \rightarrow b \cdot \eta_{\lambda}$ define a bijection from $\mathbf{B}(\lambda)$ onto a basis of Λ_{λ} ; moreover, if $b \in \mathbf{B} - \mathbf{B}(\lambda)$, then $b \cdot \eta_{\lambda} = 0$ (cf Th.14.4.11 of [Lu]).

An equivalent statement is that

$$\bigcup_{i, n; n \geq \langle i, \lambda \rangle + 1} {}^{\sigma} \mathbf{B}_{i, n}$$

is a basis of $\sum_i U^- F_i^{\langle i, \lambda \rangle + 1}$ which follows from Th.14.3.2 and §14.4.1 of [Lu].

2.3. Lagrangian construction of Canonical bases. Recall that $B(\infty)$ is the crystal base of U^- as defined in §2.3 of [K]. By [GL], \mathbf{B}

and $B(\infty)$ are canonically identified. For $b \in B(\infty)$ the corresponding perverse sheaf is denoted by $L_{b,\Omega}$. Let $B(\infty; V)$ be the set of irreducible components of Λ_V . It is shown in [K] that the set $\sqcup_V B(\infty; V)$ has a crystal structure and are isomorphic to $B(\infty)$ (cf. Th.5.3.2 of [K]). We denote by $\Lambda_b \in \sqcup_{\nu \in Q_-} B(\infty; \nu)$ the corresponding element to $b \in B(\infty)$ under this isomorphism. Thus we have a one-to-one and surjective map $\psi : \sqcup_V B(\infty; V) \rightarrow \mathbf{B}$ such that $\psi(\Lambda_b) = L_{b,\Omega}$. We shall also use the notation $\Lambda_b(V)$ (resp. $L_{b,\Omega}(V)$) if Λ_b (resp. $L_{b,\Omega}$) is a subset of Λ_V (resp. belongs to \mathbf{B}_V). It is clear from the definition that $\psi(\Lambda_b(V)) = L_{b,\Omega}(V)$. By §2.2 we have $\dim u_V^- = \sharp\{L_{b,\Omega}(V)\}$, together with (b) of Th.33.1.3 of [Lu] we conclude that:

$\dim u_V^- = \sharp\{\Lambda_b(V) | \Lambda_b(V) \in \text{Irr}\Lambda_V\}$, where $\text{Irr}\Lambda_V$ is the set of irreducible components of Λ_V .

For $B \in X_V$, we define

$$\varepsilon_i(B) = \dim \text{Coker} \left(\bigoplus_{h: \text{in}(h)=i} V(\nu)_{\text{out}(h)} \xrightarrow{(B_h)} V(\nu)_i \right).$$

Let Y be a smooth algebraic variety. For any $L \in \mathcal{D}(\mathcal{Y})$, we denote by $SS(L)$ the singular support (or the characteristic variety) of L . It is known that $SS(L)$ is a closed Lagrangian subvariety of T^*Y (See [KS]).

We recall that $T^*E_{V,\Omega}$ is identified with X_V . We record the result derived above and Th.6.2.2 of [K] in the following theorem:

- Theorem 2.3.1.** 1. $\dim u_V^- = \sharp\{\Lambda_b(V) | \Lambda_b(V) \in \text{Irr}\Lambda_V\}$
 2. For any $L \in \mathcal{P}_{V,\Omega}$ the singular support $SS(L)$ is a union of irreducible components of Λ_V .
 3. For any $b \in B(\infty)$ and $i \in I$, we have

$$\Lambda_b \subset SS(L_{b,\Omega}) \subset \Lambda_b \cup \bigcup_{\varepsilon_i(b') > \varepsilon_i(b)} \Lambda_{b'}. \quad (2.2)$$

2.4. Admissible automorphisms of a quiver. An **admissible automorphism** of a the graph (I, H) consists, by definition, of a permutation $a : I \rightarrow I$ and a permutation $a : H \rightarrow H$ such that for any $h \in H$, we have $[a(h)] = a[h]$ as subsets of I and such that there is no edge joining two vertices in the same a -orbit. We assume that there is an integer $n \geq 1$ such that $a^n = 1$ both on I and on H .

An orientation Ω of our graph is said to be compatible with a if, for any $h \in \Omega$, $a(\text{in}(h)) = \text{in}(a(h))$ and $a(\text{out}(h)) = \text{out}(a(h))$. Recall that a quiver is a graph with an orientation. A quiver with a symmetry, by definition, is a quiver together with a nontrivial admissible automorphism which is compatible with the orientation.

A quiver is called affine or finite A-D-E if the underlying graph is an affine or finite A-D-E graph. All the affine or finite A-D-E quivers with symmetries have been classified in §14.1.5 and §14.1.6 of [Lu].

There is a very close connection between Cartan datum and graphs with admissible automorphisms. Given an admissible automorphism a of a finite graph (I, H) , we define I^a to be the set of a -orbits on I . For $\tilde{i}, \tilde{j} \in I^a$, we define $\tilde{i} \cdot \tilde{j} \in \mathbb{Z}$ as follows: if $\tilde{i} \neq \tilde{j}$, then $\tilde{i} \cdot \tilde{j}$ is -1 times the number of edges which join some vertex in the a -orbit \tilde{i} to some vertex in the a -orbit \tilde{j} ; $\tilde{i} \cdot \tilde{i}$ is 2 times the number of vertices in the a -orbit \tilde{i} . As shown in §13.2.9 of [Lu], (I^a, \cdot) is a Cartan datum. We shall denote the universal enveloping algebra of the corresponding Kac-Moody algebra g^a by u^a . Conversely, every Cartan datum is obtained by the construction above from graphs with admissible automorphisms (cf Prop14.1.2 of [Lu]).

By the results of §14.1.5 and §14.1.6 of [Lu], all non-symmetric affine or finite Cartan datum can be obtained from affine or finite A-D-E graphs with a nontrivial admissible automorphism by the construction above up to proportionality.

Recall \mathcal{V} is the category of finite dimensional I -graded \mathbb{C} -vector spaces. For $V \in \mathcal{V}$, we define $a(V)$ such that $a(V)_i = V_{a(i)}$. For $B \in X_V$, we define $a(B) \in X_{a(V)}$ by setting $(a(B))_h : a(V)_i \rightarrow a(V)_j$ to be $(a(B))_h = B_{a(h)}$ for any h with $\text{in}(h) = j, \text{out}(h) = i$.

Let

$$\mathbb{N}[I]^a = \{\nu \in \mathbb{N}[I] \mid \nu_i = \nu_{a(i)}, \forall i \in I\}$$

. Notice that $\mathbb{N}[I]^a$ can be identified with $\mathbb{N}[I^a]$ in a natural way. If $\nu \in \mathbb{N}[I]^a$, we shall denote, again by ν , the corresponding element in $\mathbb{N}[I^a]$.

Let U^a be the quantized algebra associated with Cartan datum (I^a, \cdot) as in §2.1. For $\nu \in \mathbb{N}[I]^a$ and a dominant integral weight λ , recall that $(U^a)_\nu^-, (u^a)_\nu^-, L(\nu, \lambda)$ are defined as in §2.1. We shall chose an orientation Ω of our graph such that it is compatible with a . Such a choice always exists by §12.1.1 of [Lu].

Lemma 2.4.1. : *Let $\psi : \Lambda_b \rightarrow L_{b,\Omega}$ be the map as defined in §2.3. Then*

$$\psi(a^{-1}\Lambda_b(V)) = a^*(L_{b,\Omega}(V))$$

Proof. Suppose

$$a^*(L_{b,\Omega}(V)) = L_{a_1(b),\Omega}(a^{-1}(V)), a^{-1}\Lambda_b(V) = \Lambda_{a_2(b)}(a^{-1}(V))$$

. Since $a : E_{V,\Omega} \rightarrow E_{a(V),\Omega}$ is an isomorphism, both a_1 and a_2 are one-to-one and onto maps. So:

$$\begin{aligned} SS(L_{a_1(b),\Omega}(a^{-1}(V))) &= SS(a^*(L_{b,\Omega}(V))) \\ &= a^{-1}(SS(L_{b,\Omega}(V))) \\ &\supset a^{-1}(\Lambda_b(V)) \\ &= \Lambda_{a_2(b)}(a^{-1}(V)) \end{aligned}$$

where in the third step we have used (3) of Th.2.3.1. So we have

$$SS(L_{b,\Omega}(V)) \supset \Lambda_{a_2 a_1^{-1}(b)}(V)$$

. Note $\epsilon_i(b)$ is bounded above by $\dim V_i$. By descending induction on $\epsilon_i(b)$, (3) of Th.2.3.1 implies that $a_2 a_1^{-1}(b) = b$ (cf Page 18 of [K]). So $a_2 = a_1$ and the lemma is proved. \square

Denote by $B(\infty; V)^a$ the set of irreducible components of Λ_V which is a -invariant as a set. Then one can show that $\sqcup_{V \in N[I]^a} B(\infty; V)^a$ has a crystal structure which is isomorphic to the crystal bases $B(\infty)^a$ of $(U^a)^-$. But we shall not use this fact in the following.

3. MODIFIED VARIETIES ASSOCIATED WITH QUIVERS [Na1]

3.1. In this section we recall the modification of the definition of varieties associated with quivers in §2.2 due to Nakajima (cf. [Na1]). The motivation of such a modification comes from the study of moduli space of instantons over A-L-E spaces.

Fix a finite graph (I, H) as in §1.3. Suppose we are given pairs of finite dimensional hermitian vector spaces V_k, W_k for each vertex k . Let us define a complex vector space $M(V, W)$ by

$$M(V, W) := (\oplus_{h \in H} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) \oplus (\oplus_k \text{Hom}(W_k, V_k) \oplus \text{Hom}(V_k, W_k))$$

. For an element of $M(V, W)$ we denote its components by B_h, i_k, j_k .

Notice $M(V, W)$ is very close to the vector space X_V defined in §2.3. In fact when $W_k = 0, \forall k \in I$, $M(V, 0) = X_V$.

As in §2.3, we can define a symplectic form $\omega_{\mathbb{C}}$ on $M(V, W)$ by:

$$\omega_{\mathbb{C}}((B, i, j), (B', i', j')) := \sum_{h \in H} \text{tr}(\epsilon(h) B_h B'_h) + \sum_k \text{tr}(i_k i'_k - i'_k j_k)$$

where $\epsilon(h) = 1$ if $h \in \Omega$, $\epsilon(h) = -1$ if $h \in \bar{\Omega}$. The symplectic vector space $M(V, W)$ decomposes into the sum $M(V, W) = M_{\Omega} \oplus M_{\bar{\Omega}}$ or Lagrangian subspaces:

$$\begin{aligned} M_{\Omega} &:= (\oplus_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) \oplus (\oplus_k \text{Hom}(W_k, V_k) \\ M_{\bar{\Omega}} &:= (\oplus_{h \in \bar{\Omega}} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) \oplus (\oplus_k \text{Hom}(V_k, W_k) \end{aligned}$$

We can consider $M_{\bar{\Omega}}$ as a dual space of M_{Ω} via $\omega_{\mathbb{C}}$, and $M(V, W)$ as the cotangent bundle of M_{Ω} .

Since V and W are hermitian vector spaces, $\text{Hom}(V, W)$ has a hermitian inner product defined by $(f, g) = \text{tr}(fg^*)$, where g^* is the hermitian adjoint. We introduce a natural hermitian inner product on $M(V, W)$ induced from the ones on V and W in this way. We introduce a quaternion module structure on $M(V, W)$ by the original complex structure I together with a new complex structure J by the formula $J(m, m') = (-m'^*, m^*)$ for $(m, m') \in M_{\Omega} \oplus M_{\bar{\Omega}}$.

The group $G_V = \prod U(V_k)$ acts on $M(V, W)$ by

$$(B_h, i_k, j_k) \rightarrow (g_{in(h)} B_h g_{out(h)}^{-1}, g_k i_k, j_k g_k^{-1})$$

, preserving the hermitian inner product and the \mathbb{H} -module structure. Let μ be the corresponding Hyperkähler moment map (cf.[HKLR]). Denote by $\mu_{\mathbb{R}}, \mu_{\mathbb{C}}$ its real and complex components. Explicit forms of $\mu_{\mathbb{R}}, \mu_{\mathbb{C}}$ can be found in §2 of [Na1]. Let g_V be the Lie algebra of G_V and let $Z_V \subset g_V$ denote the center. Choose an element $\zeta = (\zeta_{\mathbb{R}}, 0) \in Z_V \oplus (Z_V \otimes \mathbb{C})$ with $\zeta_{\mathbb{R}} = ((i/2)\zeta_1, \dots, (i/2)\zeta_n)$ satisfies the following condition:

$$\zeta_i > 0, \forall i$$

. We define a Hyperkähler quotient $M_{\zeta}(V, W)$ of $M(V, W)$ by G_V as follows:

$$M_{\zeta}(V, W) := \{(B, i, j) \in M(V, W) | \mu(B, i, j) = -\zeta\} / G_V$$

According to the results of [Na1], $M_{\zeta}(V, W)$ is a smooth Hyperkähler manifold.

When the underlying graph is an affine or finite A-D-E graph, $M_{\zeta}(V, W)$ is a certain deformation of singularities of the moduli space of instantons over A-L-E spaces with unitary gauge group (cf.Prop.2.15 of [Na1]). In [Na2], $M_{\zeta}(V, W)$ is also referred to as the moduli space of certain torsion free sheaves.

We will be interested in certain Lagrangian subvarieties of $M_{\zeta}(V, W)$. Let us define:

$$\Lambda_{V, W} = \Lambda_V \times \oplus_k \text{Hom}(V_k, W_k)$$

where Λ_V is the variety defined in definition 2.2.1. Let us define the set of stable points by:

$$H^s = \{m \in \mu_{\mathbb{C}}^{-1}(0) | G_V^{\mathbb{C}}.m \cap \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}}) \neq \emptyset\}$$

. An alternative algebraic characterization of H^s is given in Prop.3.5 of [Na1].

The Lagrangian subvariety of $M_\zeta(V, W)$ is defined by:

$$\mathcal{L}(V, W) := \Lambda_{V, W} \cap H^s / G_V^\mathbb{C}$$

(cf. Remark 5.11 of [Na1]).

Let $\text{Irr}\mathcal{L}(V, W)$ (resp. $\text{Irr}\Lambda_{V, W}$) be the set of irreducible components of $\mathcal{L}(V, W)$ (resp. $\Lambda_{V, W}$). Since $H^s \rightarrow M_\zeta(V, W)$ is a principal $G_V^\mathbb{C}$ bundle, $\text{Irr}\mathcal{L}(V, W)$ can be identified with

$$\{Y \in \text{Irr}\Lambda_{V, W} \mid Y \cap H^s \neq \emptyset\}$$

. This can be also identified with

$$\{Y \in \text{Irr}\Lambda_V \mid Y \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\}$$

. Recall \mathcal{K}_Ω is the Grothendieck group defined in §1.2. Let $\mathcal{K}_\Omega^1(W)$ be the $\mathbb{Z}[v, v^{-1}]$ submodule of \mathcal{K}_Ω generated by complexes L in \mathcal{Q}_Ω such that

$$SS(L) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s = \emptyset$$

. The following lemma is proved in essentially the same way as the proof of lemma 11.5 of [Na1]:

Lemma 3.1.1. $\mathcal{K}_\Omega^1(W) \otimes \mathbb{Q}(v)$ is a left ideal in $\mathcal{K}_\Omega \otimes \mathbb{Q}(v)$ containing $F_i^{W_i+1}$.

Lemma 3.1.1 implies

$$\{L_{b, \omega}(V) \mid SS(L_{b, \omega}(V)) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s = \emptyset\} \supset \cup_{i \in I} {}^\sigma \mathbf{B}_V; i; \geq W_{i+1}$$

by Th.14.3.2 and §14.4.1 of [Lu]. So:

$$\{L_{b, \omega}(V) \mid SS(L_{b, \omega}(V)) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\} \subset \mathbf{B}_V(\dim W)$$

where $\mathbf{B}_V(W) := \mathbf{B}(\dim W) \cap {}^\sigma \mathbf{B}_V$ and we have identified $\dim W$ with the dominant integral weights of U as in §2.2.

Lemma 3.1.2. The map ψ from $\{\Lambda_b(V) \in \text{Irr}\Lambda_V \mid \Lambda_b(V) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\}$ to $\{L_{b, \omega}(V) \mid SS(L_{b, \omega}(V)) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\}$ is one-to-one and surjective. Moreover, $\{L_{b, \omega}(V) \mid SS(L_{b, \omega}(V)) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\} = \mathbf{B}_V(\dim W)$.

Proof. By Th.2.3.1,

$$SS(L_{b, \omega}(V)) \supset \Lambda_b(V)$$

, hence the map ψ as in Lemma 3.1.2 is well defined and is clearly one-to-one by definition. To show that ψ is surjective, we just have to show that:

$$\# \{ \Lambda_b(V) \in \text{Irr}\Lambda_V \mid \Lambda_b(V) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset \} \geq \quad (3.1)$$

$$\# \{ L_{b, \omega}(V) \mid SS(L_{b, \omega}(V)) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset \}. \quad (3.2)$$

By the remarks after Lemma 3.1.1, we have:

$$\sharp \{L_{b,\omega}(V) | SS(L_{b,\omega}(V)) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\} \leq \quad (3.3)$$

$$\sharp \mathbf{B}_V(\dim W). \quad (3.4)$$

On the other hand, let us prove that:

$$\sharp \{\Lambda_b(V) \in \text{Irr} \Lambda_V | \Lambda_b(V) \times \oplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset\} = \sharp \{\text{Irr} \mathcal{L}(V, W)\} \quad (3.5)$$

Notice that when the quiver is affine or finite A-D-E, (3.5) is (2) of Prop.10.15 in [Na1] which follows from (3) of 4.16 in [L1]. The exactly same proof with possible change of notations applies to the general case by using (0) of Th.2.3.1 which is a generalization of (3) of 4.16 in [L1]. Combine the above with (d) of Th.33.1.3. and Th.14.4.11 of [Lu], we have:

$$\sharp \{\text{Irr} \mathcal{L}(V, W)\} = \sharp \mathbf{B}_V(\dim W)$$

. It follows that all the \leq or \geq above are actually $=$ and the lemma is proved. \square

Let us denote by Λ_W the simple integrable U module with dominant integral weight $\dim W$. By Lemma 3.1.1, there is a natural surjective U^- -linear map

$$\Pi : \Lambda_W \rightarrow \mathcal{K}_\Omega \otimes \mathbb{Q}(v) / \mathcal{K}_\Omega^1(W) \otimes \mathbb{Q}(v)$$

. The dimension of the weight space (with weight $W - V$) $\Lambda_W(V)$ is $\sharp \mathbf{B}_V(\dim W)$, which is the same as the corresponding weight space (with weight $W - V$) of $\mathcal{K}_\Omega \otimes \mathbb{Q}(v) / \mathcal{K}_\Omega^1(W) \otimes \mathbb{Q}(v)$ by Lemma 3.1.2. It follow that Π is an isomorphism. Let us record this result in the following theorem:

Theorem 3.1.3. *Π is an isomorphism. The set $\text{Irr} \mathcal{L}(V, W)$ parameterizes a basis of the weight space $\Lambda_W(V)$.*

In the case of finite A-D-E quivers, Th.3.1.3 is proved in §11 of [Na1] by a different method.

3.2. Quiver varieties with symmetries. Assume that (I, H) is a finite graph with a nontrivial admissible automorphism a . We shall fix an orientation Ω which is compatible with a . Such an orientation always exists by §12.1.1 of [Lu].

Recall u^a denotes the universal enveloping algebra of the corresponding non-symmetric Kac-Moody algebra (cf. §1.4). For $W \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$, we denote by $L^a(W)$ the simple integrable module of u^a with integral dominant weight $\dim W$ and $L^a(V, W)$ the subspace of $L^a(W)$ as defined for any simple integrable module at the end of §2.1. Here $V \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$.

The natural action of a on $M(V, W)$, defined similarly as the action of a on X_V in §1.4, induces an isomorphism, still denoted by $a : M_\zeta(V, W) \rightarrow M_{a(\zeta)}(a(V), a(W))$, where $a(\zeta) = (a(\zeta_{\mathbb{R}}), 0)$ and $(a(\zeta_{\mathbb{R}}))_i = (\zeta_{\mathbb{R}})_{a(i)}$. Let us fix ζ as in §3.1 with $a(\zeta) = \zeta$. It is easy to see that only in the case $W, V \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$ that a maps $M_\zeta(V, W)$ to itself.

Theorem 3.2.1. *Let $W, V \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$. Then:*

$$\dim L^a(V, W) = \#\{Y \in \text{Irr}\mathcal{L}(V, W) | a.Y = Y\}$$

Proof. By Lemma 2.4.1 and Lemma 3.1.2, we have:

$$\begin{aligned} \#\{Y \in \text{Irr}\mathcal{L}(V, W) | a.Y = Y\} &= \\ \#\{L_{b,\Omega}(V) | SS(L_{b,\Omega}(V)) \times \bigoplus_k \text{Hom}(V_k, W_k) \cap H^s \neq \emptyset, a^*L_{b,\Omega}(V) = L_{b,\Omega}(V)\} \\ &= \#\{L_{b,\Omega}(V) \in \mathbf{B}_V(W) | a^*L_{b,\Omega}(V) = L_{b,\Omega}(V)\} \end{aligned}$$

By Th.14.4.9 and (d) of Th.13.1.3 of [Lu],

$$\#\{L_{b,\Omega}(V) \in \mathbf{B}_V(W) | a^*L_{b,\Omega}(V) = L_{b,\Omega}(V)\} = \dim L^a(V, W)$$

. Combining this with the previous identities, the theorem is proved. \square

We'd like to describe an application of Th.3.2.1 by constructing functions with certain modular properties from the geometry of $M_\zeta(V, W)$. Suppose the underlying graph is an affine A-D-E graph with a symmetry a . For simplicity we shall restrict to the case where (I^a, \cdot) is the Cartan datum of non-twisted affine Kac-Moody algebra.

Let $V, W \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$. On the Dykin diagram associated with (I^a, \cdot) , assume \tilde{i} corresponds to the label α_i on Page 54 of [Kac]. We identify W with the dominant integral weight of g^a by identifying $\tilde{i} \in I^a$ with the fundamental weight Λ_i of g^a (cf. §12 of [Kac]). Define V_0 to be V_r for any $r \in \tilde{0}$. Let m_W be defined as in formula 12.7.5. on Page 226 of [Kac].

Let $k(V, W)$ be the complex dimension of $M_\zeta(V, W)$. Define

$$L(W) := \bigoplus_{V, k=k(V, W)} H^k(M_\zeta(V, W), \mathbb{Q}),$$

$$L(V, W) := H^{k(V, W)}(M_\zeta(V, W), \mathbb{Q})$$

. By Th.10.16 of [Na1], $L(W)$ is the simple integrable module of u with integral dominant weight W .

Let $V, W \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$, define a map $L'_0 : L(V, W) \rightarrow L(V, W)$ by $L'_0 x = (m_W + V_0)x, \forall x \in L(V, W)$.

If V or W is not an element of $\mathbb{N}[I]^a$, we define $L'_0 : L(V, W) \rightarrow L(V, W)$ to be identity map.

The isomorphism $a : M_\zeta(V, W) \rightarrow M_{a(\zeta)}(a(V), a(W))$ induces a natural map :

$$a^* : H^k(M_\zeta(a(V), a(W)), \mathbb{Q}) \rightarrow H^k(M_\zeta(V, W), \mathbb{Q})$$

. Let q be an indeterminate. We define a weighted map

$$a^*(q) := a^* q^{L'_0} : H^k(M_\zeta(a(V), a(W)), \mathbb{Q}) \rightarrow H^k(M_\zeta(V, W), \mathbb{Q})$$

. Define

$$Ch^a(W) := Tr(a^*(q))$$

where $a^*(q)$ is an operator from $L(W)$ to itself. It is obvious from the definition that $Ch^a(W) = 0$ unless $W \in \mathbb{N}[I]^a \simeq \mathbb{N}[I^a]$. Moreover, the only nonzero contribution to $Ch^a(W)$ comes from the term $a^*(q) : H^k(M_\zeta(V, W), \mathbb{Q}) \rightarrow H^k(M_\zeta(V, W), \mathbb{Q})$ with $W, V \in \mathbb{N}[I]^a$. By Cor.5.5 of [Na1] and Th.3.2.1, The trace of the map $a^*(q) : H^k(M_\zeta(V, W), \mathbb{Q}) \rightarrow H^k(M_\zeta(V, W), \mathbb{Q})$ is equal to $q^{m_W + V_0}$ times $\dim L^a(V, W)$. It is easy to check that L'_0 is the same as the operator $L_0^g - \frac{1}{24}c(k)$ in formula 13.8.3 on Page 264 of [Kac] on $L^a(V, W)$. Hence $Ch^a(W) = \chi_W(\tau, 0, 0)$, where $\chi_W(h, 0, 0)$ is defined in formula 13.8.3 on Page 264 of [Kac], and $q = \exp(2\pi i\tau)$. Let us record this result in the following corollary:

Corollary 3.2.2. $Ch^a(W) = \chi_W(\tau, 0, 0)$, where $\chi_W(\tau, 0, 0)$ is defined in formula 13.8.3 on Page 264 of [Kac].

By the above corollary, the functions $Ch^a(W)$ have certain modular properties as in Th.13.8 of [Kac].

In the case of twisted affine algebras, one can define similar functions as $Ch^a(W)$ from the geometry of $M_\zeta(V, W)$ such that it coincides with the normalized character defined on Page 264 of [Kac] when $z = \mu = 0$. Such functions also have certain modular properties as shown in Th.13.9 of [Kac].

4. QUESTIONS

The proof of Th.3.2.1. is based on the deep theory of canonical bases of quantized Kac-Moody algebras. It will be very interesting to proceed further to construct a natural analogue of Hecke operators in [Na1] on the set of constructible functions on $\mathcal{L}^a(V, W)$ such that these operators define a representation of u^a . Th.3.2.1 suggests that it is possible to do so but we don't know the answer.

In [L1], an explicit description of canonical bases for symmetric affine Kac-Moody algebras are given. It will be interesting to use this result

to give a proof of Th.2.3.1 in the affine A-D-E case. This will avoid the use of Th.2.3.1 in the proof of Cor.3.2.2.

As mentioned in the introduction, Cor.3.2.2. suggests that it should be possible to take into account the actions of the nontrivial admissible automorphism in the formulation of twisted N=4 SYM theory on certain A-L-E spaces such that the partition functions are identified with the functions in Cor.3.2.2. Once this is done, Cor.3.2.2. together the results of [Kac] can be seen as further evidence in favor of S-duality.

On Page 93 of [VW] , it is asked if one can find for each rational conformal field (RCFT) theory a possibly non-compact four-manifold whose N=4 twisted partition function gives the characters of that RCFT. By Cor.3.2.2, it seems that the natural candidate in the case of Wess-Zumino-Witten model based on non-simply-laced group should be certain A-L-E spaces together with a nontrivial admissible automorphism. However it is not clear to us what exactly the four-manifold should be.

S-duality of N=4 SYM theory in 4-dimension can be derived by dimensional reduction from the duality between heterotic string compactified on 4-dimensional torus and type IIA string compactified on K3 surface ¹. The work of [Na1] has been interpreted in this string-string duality context in [HM] and [V]. It will be interesting to see if one can take into account the action of a non-trivial admissible automorphism in the formulation of heterotic/ type IIA duality so at least part of Cor.3.2.2. follows from such a formulation.

Finally, let us note that in the theory of duality between F theory compactified on Calabi-Yau manifolds and heterotic string compactified on K3 surfaces, the Dynkin diagram automorphisms corresponding to the non-trivial admissible automorphism of finite A-D-E graph have been used to explain the appearance of non-simply laced gauge group in heterotic string (cf. [MV] and [A]). Is it possible to make connections with the results of this paper by the web of string dualities?

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¹ However there is a constraint on the possible rank of the gauge group allowed, cf. [LLL]

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